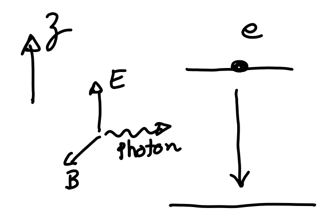
Stimulated optical emission

Atomic dipole d_{mn} interacts with the oscillating electric field of the photon. The dipole matrix element is:

$$W_{mn} = \langle m | \underbrace{e\hat{r}}_{\text{Atomic dipole}} \cdot \underbrace{\vec{E}}_{\text{photon's electric field } = \vec{E}_0 \cos(\omega t)} | n \rangle = e \langle m | \hat{r} \, |E_0| \cos \omega t \, | n \rangle$$



$$W_{mn} = e \underbrace{\langle m | \hat{r} | n \rangle}_{r_{mn}} |E_0| \cos \omega t$$

$$= e r_{mn} |E_0| \cos \omega t$$

$$\vec{E} = \vec{E}_z \Rightarrow r_{mn} = z_{mn} \Rightarrow W_{mn} = e z_{mn} |E_0| \cos \omega t$$

In deriving Fermi's Golden rule we had:

$$a_m(t) = \frac{1}{i\hbar} \int_0^t W_{mn} e^{i\omega_{mn} t'} dt' \; ; \quad \omega_{mn} = \omega_m - \omega_n$$

$$= \frac{1}{i\hbar} e |E_0| z_{mn} \int_0^t \cos \omega t' \, e^{i\omega_{mn} t'} dt'$$

$$= \frac{e |E_0| z_{mn}}{i 2\hbar} \int_0^t \left(e^{i\omega t'} + e^{-i\omega t'} \right) e^{i\omega_{mn} t'} dt'$$

$$= -\frac{e |E_0| z_{mn}}{2\hbar} \left(\frac{e^{i(\omega + \omega_{mn})t} - 1}{\omega + \omega_{mn}} + \frac{e^{-i(\omega - \omega_{mn})t} - 1}{\omega - \omega_{mn}} \right)$$

For $\omega \sim \omega_{mn}$, we have: $\omega+\omega_{mn} \gg \omega-\omega_{mn}$, and the first term is negligible:

$$a_m(t) \sim -\frac{e|E_0| z_{mn}}{2\hbar} \frac{e^{-i(\omega-\omega_{mn})t}-1}{\omega-\omega_{mn}}$$

$$= -\frac{e|E_0| z_{mn}}{2\hbar} \frac{e^{-i(\omega-\omega_{mn})t}}{\omega-\omega_{mn}} \left(e^{-i(\omega-\omega_{mn})t/2} - e^{i(\omega-\omega_{mn})t/2}\right)$$

$$= -\frac{e|E_0| z_{mn}}{2\hbar} \frac{e^{-i(\omega-\omega_{mn})t}}{\omega-\omega_{mn}} \left(-2i \sin \frac{(\omega-\omega_{mn})t}{2}\right)$$

Hence the transition probability is:

$$|a_m(t)|^2 = \left(\frac{e|E_0|}{\hbar}\right)^2 |z_{mn}|^2 \frac{\sin^2 \frac{(\omega-\omega_{mn})t}{2}}{(\omega-\omega_{mn})^2}$$

The transition into a continuum line shape is:

$$|a_m(t)|^2 = \left(\frac{e|E_0|}{\hbar}\right)^2 |z_{mn}|^2 \int_{-\infty}^{\infty} \frac{\sin^2 \frac{(\omega-\omega_{mn})t}{2}}{(\omega-\omega_{mn})^2} d\omega$$

Since: $\int_{-\infty}^{\infty} \frac{\sin^2(ax)}{x^2} dx = a\pi$. Use $a = t/2$, $x = \omega - \omega_{mn}$ $\Rightarrow$

$$|a_m(t)|^2 = \left(\frac{e|E_0|}{\hbar}\right)^2 |z_{mn}|^2 \frac{\pi t}{2}$$

for the light density, we had: $U(\omega) = \frac{1}{2}\epsilon_0 |E_0|^2 \Rightarrow$

$$|E_0|^2 = \frac{2U(\omega)}{\epsilon_0} \Rightarrow$$

$$|a_m(t)|^2 = \frac{e^2}{\hbar^2} \frac{U(\omega)}{\epsilon_0} |\beta_{mn}|^2 \pi t$$

The transition rate is:

$$\frac{d}{dt} |a_m(t)|^2 = \frac{\pi e^2}{\epsilon_0 \hbar^2} |\beta_{mn}|^2 U(\omega)$$

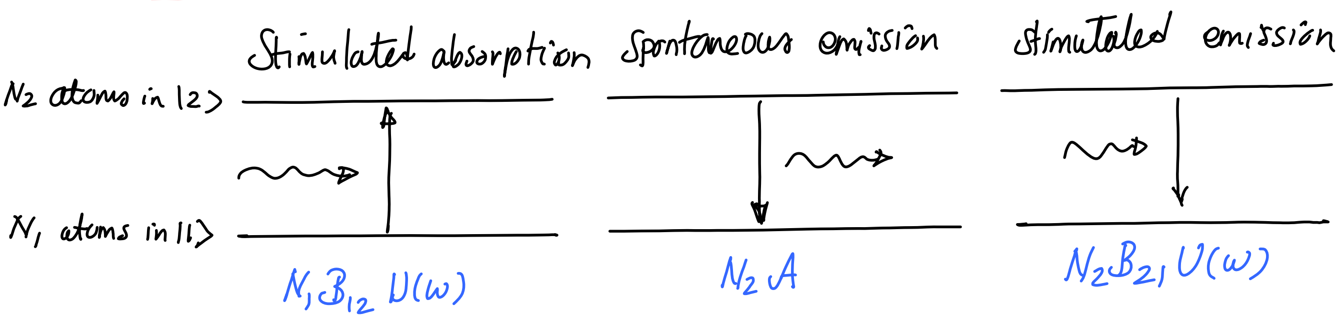
Averaging over the three directions: $|\beta_{mn}|^2 = \frac{1}{3} |r_{mn}|^2$

$$\frac{d}{dt} |a_m(t)|^2 = \frac{\pi e^2}{3\epsilon_0 \hbar^2} |r_{mn}|^2 U(\omega) = B\, U(\omega)$$

Probability of transition from $|m\rangle \rightarrow |n\rangle$ per unit time stimulated by electromagnetic radiation is:

$$\boxed{B_{mn} U(\omega) = \frac{\pi e^2}{3\epsilon_0 \hbar^2} |\langle m|\hat{r}|n\rangle|^2 U(\omega)}$$

The Einstein A and B coefficients



Under thermal equilibrium:

transition from $|2\rangle \rightarrow |1\rangle$ = $|1\rangle \rightarrow |2\rangle$

$$N_2 \left(B_{21} U(\omega) + A \right) = N_1 \left(B_{12} U(\omega) \right)$$

At thermal equilibrium: $\frac{N_1}{N_2} = e^{\hbar\omega/k_BT}$ i.e. Boltzmann distribution.

$$\Rightarrow \quad B_{21} U(\omega) + A = \frac{N_1}{N_2} B_{12} U(\omega)$$

$$\frac{B_{21}}{B_{12}} + \frac{A}{B_{12} U(\omega)} = \frac{N_1}{N_2} = e^{\hbar\omega/k_BT}$$

Substitute $U(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_BT} - 1}$ for black-body radiation:

$$\frac{B_{21}}{B_{12}} + \frac{A}{B_{12}} \frac{\pi^2 c^3}{\hbar\omega^3} (e^{\hbar\omega/k_BT} - 1) = e^{\hbar\omega/k_BT}$$

$$\underbrace{\frac{B_{21}}{B_{12}} - \frac{A}{B_{12}} \frac{\pi^2 c^3}{\hbar\omega^3}}_{=0} + \underbrace{\left(\frac{A}{B_{12}} \frac{\pi^2 c^3}{\hbar\omega^3} - 1\right)}_{=0} e^{\hbar\omega/k_BT} = 0$$

$$\Rightarrow \begin{cases} \frac{A}{B_{12}} \frac{\pi^2 c^3}{\hbar\omega^3} = 1 \\ \frac{B_{21}}{B_{12}} = 1 \end{cases} \Rightarrow \boxed{\begin{array}{l} A = \frac{\hbar\omega^3}{\pi^2 c^3} B_{12} \\ B_{12} = B_{21} \end{array}}$$

We already calculated B:

$$B = \frac{\pi e^2}{3\epsilon_0 \hbar^2} |\langle j | \hat{r} | k \rangle|^2 \quad \Rightarrow$$

$$\frac{1}{\tau_{sp}} = A = \frac{\hbar\omega^3}{\pi^2 c^3} B = \frac{e^2 \omega^3 n_r^3}{3\pi\epsilon_0 \hbar c^3} |\langle j | \hat{r} | k \rangle|^2$$

n_r: refractive index of the medium.

Example Spontaneous emission coefficient A in H atom

from $|2p\rangle \to |1s\rangle$:

$\langle 1s|\hat{r}|2p\rangle \sim a_B = 0.053$ Bohr radius

Emission wavelength is : $\lambda = 122\ nm \Rightarrow \omega = ck = c\frac{2\pi}{\lambda}$

$$A = \frac{(2\pi)^3}{\lambda^3}\frac{e^2}{3\pi\epsilon_0\hbar}a_B^2 = 1.12\times10^9\ s^{-1} = \frac{1}{\tau_{sp}}$$

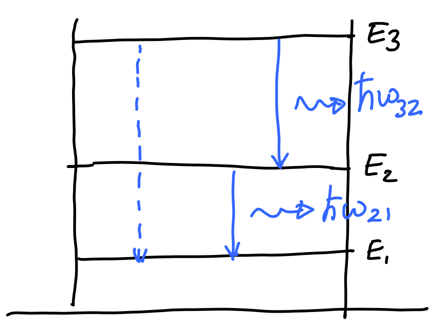
$$\Rightarrow \tau_{sp} = \frac{1}{1.12\times10^9} = 0.89\ ns$$

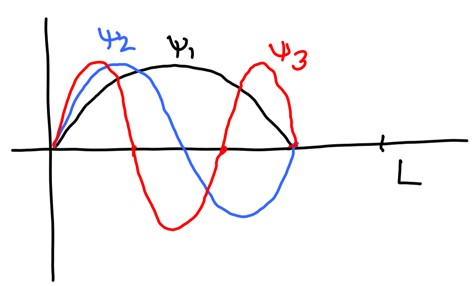
Dipole selection rules for optical transitions

$$\left.\begin{aligned}\langle even|r|even\rangle &= \int_{-\infty}^{\infty}(even)\,r\,(even) = 0\\ \langle odd|r|odd\rangle &= 0\end{aligned}\right\}\text{ No transition}$$

$$\left.\begin{aligned}\langle odd|r|even\rangle &\neq 0\\ \langle even|r|odd\rangle &\neq 0\end{aligned}\right\}\to\text{ radiation} \to \text{transition}$$

Example Spontaneous emission in infinite potential well with $L = 12.3$ nm





$$\psi_n = |n\rangle = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad n = 1, 2, \cdots$$

n odd $\rightarrow$ even parity

n even $\rightarrow$ odd parity

$$E = \frac{\hbar^2 k^2}{2m_e} = \frac{\hbar^2}{2m_e}\left(\frac{2\pi}{\lambda}\right)^2 = \frac{2\pi^2\hbar^2}{m_e}\frac{1}{\lambda^2} \Rightarrow \lambda_e^{(nm)} = \frac{1.23}{(E(ev))^{1/2}}$$

$$\frac{\lambda_1}{2} = L \rightarrow \lambda_1 = 2L \rightarrow E_1 = \left(\frac{1.23}{2\times 12.3}\right)^2 = \left(\frac{1}{20}\right)^2 = 2.5 \text{ mev}$$

$$\lambda_2 = L \rightarrow E_2 = \left(\frac{1.23}{12.3}\right)^2 = \left(\frac{1}{10}\right)^2 = 10 \text{ mev}$$

$$E_{Photon} = \hbar\omega = \hbar c k = \frac{\hbar c 2\pi}{\lambda_{Photon}} \Rightarrow \lambda_{Photon}^{(\mu m)} = \frac{1.24}{E(ev)}$$

$$\lambda_{21}^{Photon} = \frac{1.24}{E_2 - E_1} = \frac{1.24}{(10 - 2.5)\times 10^{-3}} = 165.3\ \mu m$$

Dipole matrix element

$$x_{jk} = \langle j | x | k \rangle = \frac{2}{L}\int_0^L \sin\frac{j\pi x}{L}\, x \sin\frac{k\pi x}{L}$$

$2\sin x \sin y = \cos(x-y) - \cos(x+y)$

$$= \frac{1}{L}\int_0^L x\left[\cos\frac{(j-k)\pi x}{L} - \cos\frac{(j+k)\pi x}{L}\right]$$

Use; $$\int_0^L x \cos\frac{m\pi x}{L}\, dx = \frac{L}{m\pi} x \sin\frac{m\pi x}{L}\Big|_0^L - \frac{L}{m\pi}\int_0^L \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \left[\left(\frac{L}{m\pi}\right)^2 \cos\left(\frac{m\pi x}{L}\right)\right]_0^L$$

$$= \begin{cases} \left(\frac{L}{m\pi}\right)^2 (1-1) = 0 & m = \text{even} \\ \left(\frac{L}{m\pi}\right)^2 (-1-1) = \frac{-2L^2}{m^2\pi^2} & m = \text{odd} \\ \int_0^L x\, dx = \frac{L^2}{2} & m = 0 \end{cases}$$

m is even if $|j\rangle$ and $|k\rangle$ have <u>same</u> parity $\rightarrow x_{jk}=0$

m is odd if they are of <u>different</u> parity $\rightarrow x_{jk}\neq 0$ ✓

$m=0$ if $j=k$.

$\psi_2 \rightarrow \psi_1$ transition allowed ✓

$\psi_3 \rightarrow \psi_1$ No transition ✗

$\psi_3 \rightarrow \psi_2$ ✓

$\psi_4 \rightarrow \psi_2$ ✗

<u>for $\psi_2 \rightarrow \psi_1$</u>

$$x_{12}=\frac{1}{L}\left\{\frac{-2L^2}{(2-1)^2\pi^2}-\frac{-2L^2}{(2+1)^2\pi^2}\right\}=\frac{-16L}{9\pi^2}$$

$$A=\frac{e^2\omega^3 n_r^3}{3\pi\epsilon_0\hbar c^3}|\langle j|x|k\rangle|^2 \; ; \; n_r=1, \quad \omega=2\pi f=\frac{2\pi c}{\lambda}\rightarrow$$

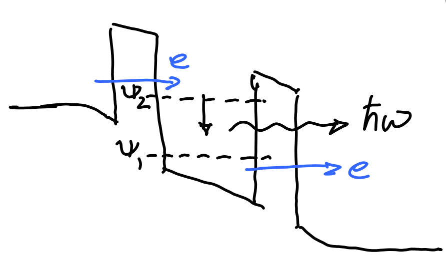
$$=\frac{7.235\times10^{17}}{\lambda^3_{photon}}|x_{12}|^2$$

$$\rightarrow \tau_{sp}=\frac{1}{A}=\frac{\lambda^3_{photon}}{7.235\times10^{17}}\frac{1}{|x_{12}|^2}=\frac{(1.65\times10^{-5})^3}{7.235\times10^{17}}\frac{81\pi^4}{256\,(12.3)^2}$$

$$=1.26\text{ ms}$$

<u>For the GaAs QW with same $L=12.3$ nm :</u>

$$\begin{cases} m_e^*=0.07m_e \\ n_r=3.3 \end{cases}$$



used for GaAs/AlGaAs Laser. $(\lambda\sim 11.5\,\mu m)$

$$\rightarrow \tau_{sp}=1.2\times10^{-8}\,s$$

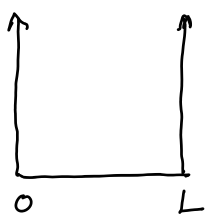
chapter 10

Time-independent perturbation

$$V(x) \longrightarrow \psi(x) \quad \text{known}$$

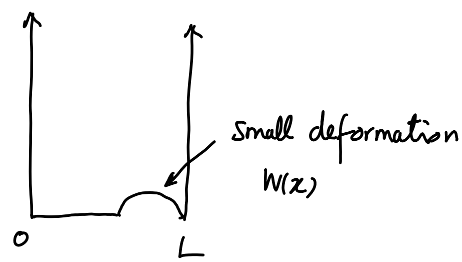
$$V'(x) \longrightarrow ?$$

Example



$\rightarrow \psi_n(x) = \sqrt{\frac{2}{L}} \sin k_n x \qquad k_n = \frac{n\pi}{L}, \; n = 1, 2, \ldots$

$$E_n = \frac{\hbar^2 k_n^2}{2m}$$



$\longrightarrow$ new eigenstates ψ_n ?

new eigenvalues E_n ?

One approach is the time-independent perturbation.

Time independent nondegenerate perturbation

$$H = H^{(0)} + W$$

$H^{(0)}$: unperturbed $\qquad$ W: perturbation

$H^{(0)} \psi_m^{(0)} = E_m^{(0)} \psi_m^{(0)}$ known

$H\psi_n = E_n \psi_n \qquad H = H_0 + W \qquad$ w is small. So we can expand ψ & E versus the unperturbed quantities:

$$\begin{cases} \psi = \psi^{(0)} + \lambda \psi^{(1)} + \lambda^2 \psi^{(2)} + \cdots \\ E = E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \cdots \end{cases}$$

Here $\lambda = 1$ and is dummy to keep track of the order of the terms.

$$H\psi = E\psi$$

$$(\hat{H}^{(0)} + \lambda \hat{W})(\psi^{(0)} + \lambda\psi^{(1)} + \lambda^2\psi^{(2)} + \cdots) = (E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \cdots)(\psi^{(0)} + \lambda\psi^{(1)} + \lambda^2\psi^{(2)} + \cdots)$$

$$H^{(0)}\psi^{(0)} + \lambda(H^{(0)}\psi^{(1)} + W\psi^{(0)}) + \lambda^2(H^{(0)}\psi^{(2)} + \hat{W}\psi^{(1)}) + \cdots =$$

$$E^{(0)}\psi^{(0)} + \lambda(E^{(0)}\psi^{(1)} + E^{(1)}\psi^{(0)}) + \lambda^2(E^{(0)}\psi^{(2)} + E^{(1)}\psi^{(1)} + E^{(2)}\psi^{(0)}) + \cdots$$

Equating equal orders of λ:

$$H^{(0)}\psi^{(0)} = E^{(0)}\psi^{(0)} \rightarrow (\hat{H}^{(0)} - E^{(0)})\psi^{(0)} = 0 \quad 0^{th} \text{ order solution}$$

$$H^{(0)}\psi^{(1)} + W\psi^{(0)} = E^{(0)}\psi^{(1)} + E^{(1)}\psi^{(0)} \rightarrow (\hat{H}^{(0)} - E^{(0)})\psi^{(1)} = (E^{(1)} - \hat{W})\psi^{(0)} \quad 1^{st} \text{ order Solution}$$

$$H^{(0)}\psi^{(2)} + \hat{W}\psi^{(1)} = E^{(0)}\psi^{(2)} + E^{(1)}\psi^{(1)} + E^{(2)}\psi^{(0)} \rightarrow (\hat{H}^{(0)} - E^{(0)})\psi^{(2)} = (E^{(1)} - \hat{W})\psi^{(1)} + E^{(2)}\psi^{(0)}$$

2^{nd} order Solution

When should we terminate the perturbation series?

This is not always clear and we must proceed with caution.

The first-order Correction

$\psi^{(0)}$ forms a complete set $\rightarrow$ expand each correction term in $\psi^{(0)}$.

For $\psi^{(1)}$ this gives:

$$\psi^{(1)} = \sum_n a_n^{(1)}\psi_n^{(0)}$$

where we know: $\hat{H}^{(0)}\psi_m^{(0)} = E_m^{(0)}\psi_m^{(0)}$

For the first order solution:

$$(H^{(0)} - E_m^{(0)})\psi^{(1)} = (E^{(1)} - \hat{W})\psi_m^{(0)}$$

$$(H^{(0)} - E_m^{(0)})\sum_n a_n^{(1)}\psi_n^{(0)} = (E^{(1)} - \hat{W})\psi_m^{(0)} \quad \times \int \psi_k^{(0)*}(\cdot) \Rightarrow$$

$$\int d^3r\, \psi_k^{(0)*}\, \underbrace{\hat{H}^{(0)}\sum_n a_n^{(1)}\psi_n^{(0)}}_{E_n^{(0)}\psi_n^{(0)}} - E_m^{(0)}\int d^3r\,\psi_k^{(0)*}\sum_n a_n^{(1)}\psi_n^{(0)} = \underbrace{E^{(1)}\delta_{km}}_{\langle k|m\rangle} - \underbrace{W_{km}}_{\langle k|W|m\rangle}$$

$$\int d^3r\, \psi_k^{(0)*} \sum_n a_n^{(1)} E_n^{(0)} \psi_n^{(0)} - E_m^{(0)} \int d^3r \sum_n a_n^{(1)} \psi_k^{(0)} \psi_n^{(0)} = E^{(1)} \delta_{km} - W_{km}$$

$$\underbrace{\sum_n E_n^{(0)} a_n^{(1)} \underbrace{\int \psi_k^{(0)*} \psi_n^{(0)} d^3r}_{\delta_{kn}}}_{E_k^{(0)} a_k^{(1)}} - E_m^{(0)} \underbrace{\sum_n a_n^{(1)} \underbrace{\int d^3r\, \psi_k^{(0)} \psi_n^{(0)}}_{\delta_{kn}}}_{a_k^{(1)}} = E^{(1)} \delta_{km} - W_{km}$$

$$\boxed{a_k^{(1)} \left(E_k^{(0)} - E_m^{(0)} \right) = E^{(1)} \delta_{km} - W_{km}}$$

$$W_{km} = \langle k | \hat{W} | m \rangle = \int \psi_k^{(0)*}\, \hat{W}\, \psi_m^{(0)*}\, d^3r$$

$$\boxed{\begin{array}{ll} a_k^{(1)} = \dfrac{W_{km}}{E_m^{(0)} - E_k^{(0)}} & k \neq m \\[2em] E^{(1)} = W_{mm} & k = m \end{array}}$$

→ we assumed nondegenerate so $E_m \neq E_k$ ✓

$E^{(1)}$ is the 1st order correction to energy eigenvalue.

Apply normalization to the 1st order solution to find $a_m^{(1)}$:

$\psi = \psi^{(0)} + \psi^{(1)}$ must be normalized =>

$$\langle \psi | \psi \rangle = \langle \psi_m^{(0)} + \lambda \psi^{(1)} \,|\, \psi_m^{(0)} + \lambda \psi^{(1)} \rangle$$

$$= \langle \psi_m^{(0)} + \lambda \sum_i a_i^{(1)} \psi_i^{(0)} \,|\, \psi_m^{(0)} + \lambda \sum_j a_j^{(1)} \psi_j^{(0)} \rangle$$

$$= \langle \psi_m^{(0)} | \psi_m^{(0)} \rangle + \lambda \sum_i a_i^{(1)*} \langle \psi_i^{(0)} | \psi_m^{(0)} \rangle + \lambda \sum_j a_j^{(1)} \langle \psi_m^{(0)} | \psi_j^{(0)} \rangle + \ldots$$

$$= 1 + \lambda a_m^{(1)*} + \lambda a_m^{(1)} \underbrace{+ \cdots}_{\text{neglecting higher order terms}} = 1 \quad \Rightarrow \quad a_m^{(1)} = 0$$

So the eigenvalues & eigenfunctions to 1st order are;

$$\boxed{\begin{aligned}\psi &= \psi_m^{(0)} + \sum_{k \neq m} \frac{W_{km}}{E_m^{(0)} - E_k^{(0)}} \psi_k^{(0)} \\ E &= E_m^{(0)} + W_{mm}\end{aligned}}$$